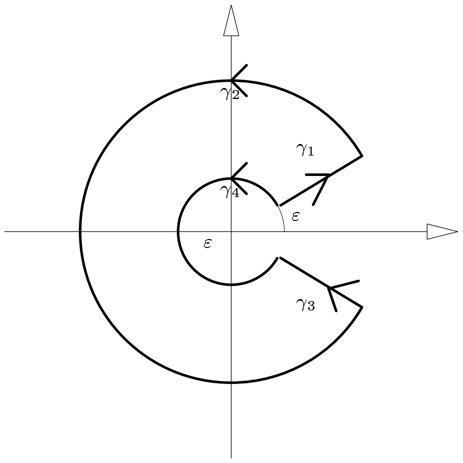
$\displaystyle\int_0^{+\infty} dx\frac{\sqrt{x}}{1+x^2}$. Consideriamo il piano complesso "tagliato" lungo l'asse $Rez>0$ ossia $D=\mathbf{C}\backslash\{Re\,z>0\}$ e la funzione $f(z)=\dfrac{\sqrt{z}}{1+z^2}$. Sia $\Gamma=\gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$. $\displaystyle\int_\Gamma dzf(z)=\sum_{i=1}^4\int_{\gamma_i}dzf(z)$.



$\gamma_1(t)=te^{i\varepsilon}$ $\varepsilon\le t\le R$. $\displaystyle\int_{\gamma_1}dzf(z)=\int_\varepsilon^R dte^{i\varepsilon}\frac{\sqrt{te^{i\varepsilon}}}{1+t^2e^{2i\varepsilon}}$ e nel limite $\varepsilon\to 0$, $R\to+\infty$ si ha

$$\lim_{\varepsilon\to 0}\lim_{R\to+\infty}\int_\varepsilon^R dte^{i\varepsilon}\frac{\sqrt{te^{i\varepsilon}}}{1+t^2e^{2i\varepsilon}}=\int_0^{+\infty}dx\frac{\sqrt{x}}{1+x^2}$$

$\gamma_2(t)=Re^{i\varphi}$ $\varepsilon\le\varphi 2\pi-\varepsilon$. $\displaystyle\int_{\gamma_2}dzf(z)=\int_\varepsilon^{2\pi-\varepsilon}d\varphi Rie^{i\varphi}\frac{\sqrt{Re^{i\varphi}}}{1+R^2e^{2i\varphi}}$. Per $R>1$ L'integrando in modulo si maggiora con $\dfrac{R^{3/2}}{R^2-1}$ e nel limite $R\to+\infty$ tende a zero

$\gamma_3(t)=-te^{i(2\pi-\varepsilon)}$ $-R\le t\le-\varepsilon$. $\displaystyle\int_{\gamma_3}dzf(z)=\int_{-R}^{-\varepsilon}-dte^{i(2\pi-\varepsilon)}\frac{\sqrt{-te^{i(2\pi-\varepsilon)}}}{1+t^2e^{2i(2\pi-\varepsilon)}}=\int_{-\varepsilon}^{-R}dte^{-i\varepsilon}\frac{\sqrt{-t}e^{i(\pi-\varepsilon/2)}}{1+t^2e^{-2i\varepsilon}}$. Il cambio di variabile $\tau=-t$ dà $\displaystyle\int_\varepsilon^R d\tau e^{-i\varepsilon}\frac{\sqrt{\tau}(-)e^{-i\varepsilon/2}}{1+\tau^2e^{-2i\varepsilon}}\cdot\xrightarrow[\varepsilon\to 0]{}\xrightarrow[R\to+\infty]{}=\int_0^{+\infty}dx\frac{\sqrt{x}}{1+x^2}$

$\gamma_4(t)=\varepsilon e^{i\varphi}$ con $\varepsilon\le\varphi\le 2\pi-\varepsilon$. Si ottiene $\displaystyle\int_{\gamma_3}dzf(z)=\int_\varepsilon^{2\pi-\varepsilon}d\varphi\varepsilon ie^{i\varepsilon}\frac{\sqrt{\varepsilon e^{i\varphi}}}{1+\varepsilon^2e^{2i\varepsilon}}\xrightarrow[\varepsilon\to 0]{}0$

La somma dei quattro limiti dà $\displaystyle 2\int_0^{+\infty}dx\frac{\sqrt{x}}{1+x^2}=2\pi i(Resf(i)+Resf(-i))$.

$$Resf(i)=\lim_{z\to i}f(z)=\lim_{z\to i}(z-i)\frac{\sqrt{i}}{(z+i)(z-i)}=\frac{\sqrt{e^{i\pi/2}}}{2i}=\frac{e^{i\pi/4}}{2i}$$

$Resf(-i)=\displaystyle\lim_{z\to -i}f(z)=\lim_{z\to i}(z+i)\frac{\sqrt{-i}}{(z+i)(z-i)}=\frac{\sqrt{e^{i3\pi/2}}}{-2i}=\frac{e^{i3\pi/4}}{-2i}$. Alla fine si ottiene $2\pi i(\dfrac{e^{i\pi/4}}{2i}-\dfrac{e^{i3\pi/4}}{2i})=\pi\sqrt{2}$.